

Perturbation method in the assessment of radiation reaction in the capture of stars by black holes

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Abstract. This work deals with the motion of a radially falling star in Schwarzschild geometry and correctly identifies radiation reaction terms by the perturbative method. The results are: i) identification of all terms up to first order in perturbations, second in trajectory deviation, and mixed terms including lowest order radiation reaction terms; ii) renormalisation of all divergent terms by the ζ Riemann and Hurwitz functions. The work implements a method previously identified by one of the authors and corrects some current misconceptions and results.

MSC: 83C10 Equations of motion 83C57 Black holes 70F05 Two-body problem

1. Introduction

The burst of gravitational waves emitted during the capture of a compact star by a supermassive black hole is of main interest for space interferometry [1], for testing general relativity in strong field and for investigating the physics of black holes. The detection of such sources requires the design of templates, which in turns require understanding of the complex orbital evolution during capture. These considerations are timely and strategic, especially if we refer to the supermassive black hole in the centre of our galaxy [2], as a source to be potentially detected in the next years.

In the last years, perturbation methods have played an increasingly important role, due to their applicability to the last phases of lifetime of binaries constituted by comparable masses before merge [3]. The implication for ground interferometry [4] has not passed unnoticed [5] and considerable efforts and resources have been and are being employed.

The understanding of orbital evolution requires understanding of radiation reaction effects. Indeed, radiation reaction, still partially outstanding problem in general relativity, like the two-body problem, is of most concern for gravitational waves detectors. Its influence is manifest in data analysis, where a phase mismatch of the templates with the signals may cause loss of detection.

In this work, we analyse a radially falling star, m , captured by a massive black hole, M , by perturbative methods. The motion is studied in strong gravity, the perturbation being based on the m/M ratio. Radial fall is an idealisation of the capture scenario, but applicable to final plunging. Furthermore, most of the radiation, and thus reaction, occurs close to the horizon where inspiral has ceased. Nevertheless, we do not make specific reference to plunging as our analysis allows any distance for the point of

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departure.

We refrain from using energy balance and adiabatic hypothesis. The former is the imposition at start, rather than a rightful outcome, of the equality of the energy radiated with the energy loosed by the system. The latter can't be evoked since the particle immediately has to react to the radiation emitted, contrarily to inspiral motion where radiation reaction time scale is larger than the orbital period^{||}. Furthermore, an analysis out of adiabacity implies that a straightforward linearisation of the phenomenon under study is not justified, and instead a careful screening of relevant terms must be adopted. Hence, neglect of higher order terms is to be avoided in absence of compelling reasons.

We study the motion of a radially falling star with a perturbative approach using Moncrief [7] gauge invariant formulation of the Regge-Wheeler [8] and Zerilli [9] equations. Incidentally, also second order perturbations formalism may be made gauge invariant [10]. Under these assumptions, we shall not be concerned with gauge issues hereafter.

Finally, second order perturbation analysis [11] demands a consistent description at first order of the energy momentum tensor. The linearisation of general relativity implies that the particle motion in unperturbed Schwarzschild generates radiation; such radiation, including quadratic radiative terms of first order, plus the corrected motion, including radiation reaction, shall generate radiation at second order. Computing gravitational radiation to second order requires the knowledge of the trajectory of the falling mass on the first order metric (Schwarzschild plus perturbations).

In the second section, it shall accordingly be developed a physical hierarchical scheme for each term constituting the geodesic, classifying terms that produce accelerations deviations from the unperturbed geodesic in Schwarzschild geometry in three types: the first depending upon the unperturbed metric evaluated on the *perturbed* trajectory; the second upon the *perturbed* metric evaluated on the unperturbed trajectory; the third upon the *perturbed* metric evaluated on the *perturbed* trajectory. In the third section, the issue of renormalisation shall be dealt for the infinite sum, on all multipoles, of finite terms that lead to divergencies.

We adopt the geodesic concept in our approach, meaning that any motion is geodesic if the underlying metric is properly defined.

Finally we are revising the work on the concept of self-force [12] and correspondences between the two methods are subject of an undergoing investigation [13].

2. The metric, the perturbation scheme and the geodesic equation

Perturbation method for analysis of radiation reaction has been previously proposed [14] - [16]. The metric is the sum of the Schwarzschild metric and the perturbations:

$$\eta_{\mu\nu} = \begin{pmatrix} f & 0 \\ 0 & -\frac{1}{f} \end{pmatrix} \quad h_{\mu\nu} = \begin{pmatrix} -fH_0 & -H_1 \\ -H_1 & -\frac{1}{f}H_2 \end{pmatrix} \quad \eta^{\mu\nu} = \begin{pmatrix} \frac{1}{f} & 0 \\ 0 & -f \end{pmatrix} \quad h^{\mu\nu} = \begin{pmatrix} -\frac{1}{f}H_0 & H_1 \\ H_1 & -fH_2 \end{pmatrix}$$

$$g_{tt} = f(1 - H_0) \quad g_{tr} = g_{rt} = -H_1 \quad g_{rr} = -\frac{1}{f}(1 + H_2)$$

^{||} Inspiral motion, especially around spinning black holes, at high eccentricity and on inclined orbits is characterised by three radiation time scales [6].

$$g^{tt} = \frac{1}{f}(1 + H_0) \quad g^{tr} = g^{rt} = -H_1 \quad g^{rr} = -f(1 - H_2)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad f = \frac{r - 2M}{r}$$

The order of the perturbation can be made explicit, i.e. $h^{(1)}, h^{(2)}$. The position of the particle $r_e = r_p + \Delta r_p$ is given by the unperturbed trajectory in the unperturbed field r_p and by several contributions, among which radiation reaction, given by the unperturbed and perturbed field, that generate a trajectory deviation Δr_p . The field is developed in Taylor series around the *real* position of the particle: $g_{\mu\nu}(r_e) = g_{\mu\nu}(r_p) + \Delta r_p (\partial g_{\mu\nu} / \partial r)_{r_p}$. The geodesic is only dependent upon radial and time coordinates:

$$\frac{d^2 r}{dt^2} = \Gamma_{rr}^t \left(\frac{dr}{dt} \right)^3 + (2\Gamma_{tr}^t - \Gamma_{rr}^r) \left(\frac{dr}{dt} \right)^2 + (\Gamma_{tt}^t - 2\Gamma_{tr}^r) \left(\frac{dr}{dt} \right) - \Gamma_{tt}^r \quad (1)$$

In absence of the weak field hypothesis, h is not limited in amplitude, but the following justifies that solely the terms in *Tab. 1* are to be retained. We suppose:

$$\frac{[h^{(1)}]^2}{\eta} \simeq \frac{h^{(2)}}{\eta} \ll \frac{h^{(1)}}{\eta} < \frac{\Delta \ddot{r}_p}{\ddot{r}_p} \quad (2)$$

The latter inequivalence is due to the twofold nature of the acceleration trajectory deviation: $\Delta \ddot{r}_p$ is the sum of two types of contributions. One is given by the Schwarzschild metric ¶, the other by the perturbations h (coupled with η):

$$\Delta \ddot{r}_p = \Delta \ddot{r}_p(\eta) + \Delta \ddot{r}_p(h) \quad (3)$$

The acceleration is dependent upon $h^{(1)}$ derivatives which are not necessarily small, especially in the last phase of the trajectory. In conclusion, the terms proportional to $h^{(1)}, \Delta r_p, \Delta \dot{r}_p, h^{(1)}$ derivatives and $\Delta r_p^2, \Delta \dot{r}_p^2, \Delta r_p \Delta \dot{r}_p$ are retained, while those to $[h^{(1)}]^2$, and $h^{(2)}$ are neglected, as the second order terms in trajectory deviation when multiplied by first order perturbations.

We write the geodesic equation in the following form:

$$\Delta \ddot{r}_p = \alpha_1 \Delta r_p + \alpha_2 \Delta \dot{r}_p + \alpha_3 \Delta r_p^2 + \alpha_4 \Delta \dot{r}_p^2 + \alpha_5 \Delta r_p \Delta \dot{r}_p + \alpha_6 + \alpha_7 \Delta r_p + \alpha_8 \Delta \dot{r}_p \quad (4)$$

The physical significance of the terms is essential⁺. The terms $\alpha_{1,2,3,4,5}$ arise from the pure Schwarzschild metric and they may be alternatively interpreted as representing the coefficients of the geodesic deviation of two particles separated on the radial axis by a Δr_p distance. In the scenario of a single falling particle, they are the coefficients representing the *unperturbed* Schwarzschild metric influence calculated on the *perturbed* particle trajectory, i.e. the *real* position ($\alpha_{1,3}$), velocity ($\alpha_{2,4}$) or both (α_5). The α_6 term is the lowest order containing the perturbations. It represents the perturbation influence calculated at the position and velocity of the particle in the *unperturbed* trajectory. Thus it does *not* represent radiation reaction, although it contributes to. The α_7 term represents the perturbation influence calculated on the *real* position of the particle in the *perturbed* trajectory, $h_{\mu\nu} \Delta r_p$. Finally, the α_8 term represents the perturbation influence calculated on the *real* velocity of the particle in

¶ Indeed, the first five terms α_{1-5} provide an acceleration \ddot{r}_p depending solely upon η , the unperturbed metric

⁺ The α_1 term correspond to the A term of [17],[18], apart of an error in the quoted publications (see appendix). The α_2 term correspond to the B term, α_6 correspond to C of [17],[18].

the *perturbed* trajectory $h_{\mu\nu}\Delta\dot{r}_p$.

It is thus a development at first order in perturbations and second order in trajectory deviation or mixed terms. *Tab. 1* lists all α terms. We emphasize that:

- The terms $\alpha_{3,4,5,7,8}$ were previously [17],[18] neglected and the remaining terms $\alpha_{1,2,6}$ do not really represent radiation reaction; ii) the terms $\alpha_{3,4,5}$ are of second order in trajectory deviation. They also do not represent radiation reaction but are kept for mathematical consistency with the hypothesis (2)*. The terms $\alpha_{7,8}$ represent the lowest order radiation reaction terms.
- Solely terms in second order of perturbations are excluded. Such choice is also compliant to the nature of the energy-momentum tensor of the second order equations. The tensor $T_{\mu\nu}$, that generates second order perturbations, must entail a geodesic stemmed from eq.(4). The linearisation of general relativity induces a stepped-up approach, in where radiation and motion assume sequentially the lead. Motion generates radiation that generates reaction, which in turns generates radiation of second order (to be dealt with squared first order radiation terms). But radiation of second order requires generation by radiation reaction at first(lowest) order, i.e. the terms $\alpha_{7,8}$.
- Radiation reaction effects, supposedly, are larger near the horizon, where most of the radiation is emitted. The leading terms of α_6 near the horizon are of the order $h^{rr}\eta_{rr,r}\dot{r}_p^2 + \eta^{tt}h_{tt,t}\dot{r}_p + h^{rr}\eta_{tt,r}$, whereas in $\alpha_{7,8}$, are of the order $h^{rr}\eta_{rr,rr}\dot{r}_p^2 + \eta_{,r}^{tt}h_{tt,t}\dot{r}_p$ and $h^{rr}\eta_{rr,r}\dot{r}_p + \eta^{tt}h_{tt,t}$ to be multiplied by Δr_p and $\Delta\dot{r}_p$, respectively. The Schwarzschild metric tensor components and derivatives, above quoted, tend to infinity and its powers, while \dot{r}_p to zero, near the horizon. Estimate of the leading terms in $\alpha_{7,8}$ justifies their inclusion.
- Second order perturbations equations, are yet unsolved in presence of a source term and the vacuum solutions are adequate for dealing the “close limit” but not the particle motion. The unavailability of the second order perturbation solutions suggest to consider the problem pragmatically using all remaining known quantities, as shown in eq.(4).

3. Black hole polar perturbations equation

Zerilli [9] found the equation for polar perturbations and studied the emitted radiation adding a source term, a freely falling test mass m into the black M . The equation is written in terms of the wavefunction ψ_l for each l-pole component, the tortoise coordinate r^* , the polar potential $V_l(r)$, the 2^l -pole source component $S_l(r, t)$:

$$\frac{d^2\psi_l(r, t)}{dr^{*2}} - \frac{d^2\psi_l(r, t)}{dt^2} - V_l(r)\psi_l(r, t) = S_l(r, t) \quad r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \quad (5)$$

$$V_l(r) = \left(1 - \frac{2M}{r}\right) \frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2Mr^2 + 18\lambda M^2r + 18M^3}{r^3(\lambda r + 3M)^2} \quad \lambda = \frac{1}{2}(l-1)(l+2) \quad (6)$$

$$S_l = \frac{\left(1 - \frac{2M}{r}\right) 4M \sqrt{(2l+1)\pi}}{(\lambda+1)(\lambda r + 3M)} \times$$

* It shall be the upcoming numerical simulation to verify the relative weight of these terms [19].

$$\left\{ r \left(1 - \frac{2M}{r} \right)^2 \delta'[r - r_p(t)] - \left(\lambda + 1 - \frac{M}{r} - \frac{6Mr}{\lambda r + 3M} \right) \delta[r - r_p(t)] \right\} \quad (7)$$

where $r_p(t)$, geodesic in *unperturbed Schwarzschild metric*, is the inverse of:

$$t = -4M \left(\frac{r}{2M} \right)^{1/2} - \frac{4M}{3} \left(\frac{r}{2M} \right)^{3/2} - 2M \ln \left[\left(\sqrt{\frac{r}{2M}} - 1 \right) \left(\sqrt{\frac{r}{2M}} + 1 \right)^{-1} \right] \quad (8)$$

The perturbations around the particle are (Regge-Wheeler gauge $H_0^l = H_2^l$):

$$H_0^l = - \frac{9M^3 + 9\lambda M^2 r + 3\lambda^2 M r^2 + \lambda^2(\lambda + 1)r^3}{r^2(\lambda r + 3M)^2} \psi + \frac{3M^2 - \lambda M r + \lambda r^2}{r(\lambda r + 3M)} \psi_{,r} + (r - 2M) \psi_{,rr} \quad (9)$$

$$H_1^l = r \psi_{,rt} - \frac{3M^2 + 3\lambda M r - \lambda r^2}{(r - 2M)(\lambda r + 3M)^2} \psi_{,t} \quad (10)$$

The *unperturbed* velocity is given by (r_0 is the test mass position at start):

$$\dot{r}_p = - \left(1 - \frac{2M}{r_p} \right) \left(\frac{2M}{r_p} - \frac{2M}{r_0} \right)^{1/2} \left(1 - \frac{2M}{r_0} \right)^{-1/2} \quad (11)$$

4. Renormalisation

In this section we reconstruct the renormalisation for α_6 and apply, for the first time, the ζ function to $\alpha_{7,8}$. Indeed, the infinite sum over the finite multipole components contributions leads to the problem of dealing infinities in the results. For ever larger l the metric perturbations tend to an asymptotic behaviour. In other words, the curves representing each metric perturbation component for each l , accumulate over the $l \rightarrow \infty$ curve. Thus the subtraction from each mode of the $l \rightarrow \infty$ leads to a convergent series. We extend the application of the Riemann ζ function for renormalisation [17] - [18] to all pertinent terms of the geodesic of *Tab. 1*. Instead, mode-sum renormalisation is planned in the near future. For $L = l + 0.5$, the wavefunction and its derivatives assume the following forms at large L or l [18], [20]‡ when averaged around the particle at r_p :

$$\begin{aligned} \bar{\psi} &\simeq 4\sqrt{2\pi m} L^{-2.5} & \bar{\psi}_{,r} &\simeq - \frac{6\sqrt{2\pi m}(r_0 - 2M)}{r_0(r_p - 2M)} L^{-2.5} & \bar{\psi}_{,rr} &\simeq \frac{4\sqrt{2\pi m}(r_0 - 2M)}{r_0(r_p - 2M)^2} L^{-0.5} \\ \bar{\psi}_{,rrr} &\simeq \frac{4\sqrt{2\pi m}(r_0 - 2M)}{r_0(r_p - 2M)^3} \left[\frac{5(r_0 - 2M)}{2r_0} + \frac{9M}{r_p} - 6 \right] L^{-0.5} \\ \bar{\psi}_{,t} &\simeq \frac{6\sqrt{2\pi m}\sqrt{r_0 - 2M}\dot{r}_p}{\sqrt{r_0}r_p} L^{-2.5} & \bar{\psi}_{,tr} &\simeq - \frac{4\sqrt{2\pi m}\sqrt{r_0 - 2M}\dot{r}_p}{\sqrt{r_0}r_p(r_p - 2M)} L^{-0.5} \\ \bar{\psi}_{,trr} &\simeq - \frac{4\sqrt{2\pi m}\sqrt{r_0 - 2M}\dot{r}_p}{\sqrt{r_0}r_p(r_p - 2M)^2} \left[\frac{5(r_0 - 2M)}{2r_0} + \frac{9M}{r_p} - 4 \right] L^{-0.5} \end{aligned}$$

‡ The derivation of such expressions, quoted from a paper in preparation by Barack and Lousto, as referred by [18] and [21], is yet unpublished.

Using the above equations, eqs.(9,10) and recasting $H_{1,t}$ as function of $\bar{\psi}$, $\bar{\psi}_{,r}$, $\bar{\psi}_{,rr}$, $\bar{\psi}_{,rrr}$, the α_6 term for large L or l around the particle is:

$$\alpha_6 = \sum_{l=0}^{\infty} \alpha_6^l \quad \alpha_6^l = \alpha_6^a L^0 + \alpha_6^b L^{-2} + \alpha_6^c L^{-4} + O(L^{-6}) \quad (12)$$

The term $\alpha_6^a L^0$ needs^{††} renormalisation [22]. The Riemann ζ function [23] and its generalisation, the Hurwitz ζ function [24], are defined by:

$$\zeta(s) = \sum_{l=1}^{\infty} (l)^{-s} \quad \zeta(s, a) = \sum_{l=0}^{\infty} (l+a)^{-s} \quad (13)$$

where in our case $a = 0.5$. Thus:

$$\zeta(s, 0.5) = \sum_{l=0}^{\infty} (l+0.5)^{-s} = 2^s \left[\sum_{l=0}^{\infty} (2l+1)^{-s} \right] \quad (14)$$

Due to the imparity of the term in braces, eq.(14) is rewritten as:

$$\zeta(s, 0.5) = 2^s \left\{ \zeta(s) - \left[\sum_{l=0}^{\infty} (2l)^{-s} \right] \right\} = 2^s (1 - 2^{-s}) \zeta(s) = (2^s - 1) \zeta(s) \quad (15)$$

Some special values of the Hurwitz functions are:

$$\zeta(-2, 0.5) = 0 \quad \zeta(0, 0.5) = 0 \quad \zeta(2, 0.5) = \frac{1}{2} \pi^2 \quad \zeta(4, 0.5) = \frac{1}{6} \pi^4 \quad (16)$$

The latter values when applied to eq.(12), give:

$$\alpha_6 = \alpha_6^a \sum_{l=0}^{\infty} (l+0.5)^0 + \alpha_6^b \sum_{l=0}^{\infty} (l+0.5)^{-2} + \alpha_6^c \sum_{l=0}^{\infty} (l+0.5)^{-4} + [0(l+0.5)^{-6}] =$$

$$\alpha_6^a \zeta(0, 0.5) + \alpha_6^b \zeta(2, 0.5) + \alpha_6^c \zeta(4, 0.5) + [0(l+0.5)^{-6}] = \frac{1}{2} \pi^2 \alpha_6^b + \frac{1}{6} \pi^4 \alpha_6^c + [0(l+0.5)^{-6}] \quad (17)$$

For the renormalisation of $\alpha_{7,8}$ terms, the expressions: $\bar{\psi}_{,tt}\bar{\psi}_{,ttt}\bar{\psi}_{,ttr}\bar{\psi}_{,trrr}\bar{\psi}_{,rrrr}$ are deducted [22] operating on the averaged wavefunctions and derivatives, and the homogeneous wave equation. The latter is recast as [14] - [15]:

$$\frac{1}{\rho^2} \frac{d^2 \psi_l(r, t)}{dr^2} - \frac{d^2 \psi_l(r, t)}{dt^2} + \frac{\rho - 1}{r \rho^2} \frac{d \psi_l(r, t)}{dr} - V_l(r) \psi_l(r, t) = 0 \quad (18)$$

where $\rho = dr^*/dr$. Deriving sequentially eq.(18), we get the ψ needed derivatives. The latter are evaluated for $L \rightarrow \infty$ and when inserted in the $\alpha_{7,8}$ terms, result into:

$$\alpha_7 = \sum_{l=0}^{\infty} \alpha_7^l \quad \alpha_7^l = \alpha_7^a L^2 + \alpha_7^b L^0 + \alpha_7^c L^{-2} + \alpha_7^d L^{-4} + O(L^{-6}) \quad (19)$$

$$\alpha_8 = \sum_{l=0}^{\infty} \alpha_8^l \quad \alpha_8^l = \alpha_8^a L^0 + \alpha_8^b L^{-2} + \alpha_8^c L^{-4} + O(L^{-6}) \quad (20)$$

Renormalisation of eqs. (19,20) leads to:

$$\alpha_7 = \frac{1}{2} \pi^2 \alpha_7^c + \frac{1}{6} \pi^4 \alpha_7^d + [0(l+0.5)^{-6}] \quad \alpha_8 = \frac{1}{2} \pi^2 \alpha_8^b + \frac{1}{6} \pi^4 \alpha_8^c + [0(l+0.5)^{-6}] \quad (21)$$

^{††} The term $\alpha_6^a L^0$ differs from b of eq.(13) in [18] which is again different from the value given in [20].

5. Conclusions

We have obtained the following results: i) calculation and determination of all terms up to first order in perturbations, second in trajectory deviation, mixed term of second order including lowest order radiation reaction terms, all contributing to the trajectory of a radially falling test mass in Schwarzschild geometry; ii) renormalisation of all divergent terms, including new ones, stemmed from the infinite sum of finite angular momentum dependent components by the ζ Riemann and Hurwitz functions; iii) correction and improvements of previously published results.

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Appendix on the geodesic equation

Three perturbations schemes concur:

$$r_e = r_p + \Delta r_p \quad g_{\mu\nu}(r_e) = g_{\mu\nu}(r_p) + \Delta r_p \left(\frac{\partial g_{\mu\nu}}{\partial r} \right)_{r_p} \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (22)$$

Hence, the development of eq.(1) leads to a lengthy computation

$$\begin{aligned} \Gamma_{rr}^t \left(\frac{dr}{dt} \right)^3 &\simeq \frac{1}{2} [g^{tt}(2g_{tr,r} - g_{rr,t}) + g^{tr}g_{rr,r}] (\dot{r}_p^3 + 3\dot{r}_p^2\Delta\dot{r}_p + 3\dot{r}_p\Delta\dot{r}_p^2) \simeq \\ &\frac{1}{2} [(\eta^{tt} + h^{tt} + \eta_{,r}^{tt}\Delta r_p + h_{,r}^{tt}\Delta r_p) (2h_{tr,r} + 2h_{tr,rr}\Delta r_p - h_{rr,t} - h_{rr,tr}\Delta r_p) + \\ &(h^{tr} + h_{,r}^{tr}\Delta r_p) (\eta_{rr,r} + h_{rr,r} + \eta_{rr,rr}\Delta r_p + h_{rr,rr}\Delta r_p)] (\dot{r}_p^3 + 3\dot{r}_p^2\Delta\dot{r}_p + 3\dot{r}_p\Delta\dot{r}_p^2) \quad (23) \end{aligned}$$

$$\begin{aligned} 2\Gamma_{tr}^t \left(\frac{dr}{dt} \right)^2 &= [g^{tt}g_{tt,r} + g^{tr}g_{rr,t}] (\dot{r}_p^2 + 2\dot{r}_p\Delta\dot{r}_p + \Delta\dot{r}_p^2) \simeq \\ &[(\eta^{tt} + h^{tt} + \eta_{,r}^{tt}\Delta r_p + h_{,r}^{tt}\Delta r_p) (\eta_{tt,r} + h_{tt,r} + \eta_{tt,rr}\Delta r_p + h_{tt,rr}\Delta r_p) + \\ &(h^{tr} + h_{,r}^{tr}\Delta r_p) (h_{rr,t} + h_{rr,tr}\Delta r_p)] (\dot{r}_p^2 + 2\dot{r}_p\Delta\dot{r}_p + \Delta\dot{r}_p^2) \quad (24) \end{aligned}$$

$$\begin{aligned} -\Gamma_{rr}^r \left(\frac{dr}{dt} \right)^2 &= -\frac{1}{2} [g^{rr}g_{rr,r} + g^{rt}(2g_{tr,r} - g_{rr,t})] (\dot{r}_p^2 + 2\dot{r}_p\Delta\dot{r}_p + \Delta\dot{r}_p^2) \\ &\simeq -\frac{1}{2} [(\eta^{rr} + h^{rr} + \eta_{,r}^{rr}\Delta r_p + h_{,r}^{rr}\Delta r_p) (\eta_{rr,r} + h_{rr,r} + \eta_{rr,rr}\Delta r_p + h_{rr,rr}\Delta r_p) + \\ &(h^{rt} + h_{,r}^{rt}\Delta r_p) (2h_{tr,r} + 2h_{tr,rr}\Delta r_p - h_{rr,t} - h_{rr,tr}\Delta r_p)] (\dot{r}_p^2 + 2\dot{r}_p\Delta\dot{r}_p + \Delta\dot{r}_p^2) \quad (25) \end{aligned}$$

$$\begin{aligned} \Gamma_{tt}^t \left(\frac{dr}{dt} \right) &= \frac{1}{2} [g^{tt}g_{tt,t} + g^{tr}(2g_{rt,t} - g_{tt,r})] (\dot{r}_p + \Delta\dot{r}_p) \simeq \\ &\frac{1}{2} [(\eta^{tt} + h^{tt} + \eta_{,r}^{tt}\Delta r_p + h_{,r}^{tt}\Delta r_p) (h_{tt,t} + h_{tt,rr}\Delta r_p) + \end{aligned} \quad (26)$$

$$(h^{tr} + h_{,r}^{tr}\Delta r_p) (2h_{rt,t} + 2h_{rt,tr}\Delta r_p - \eta_{tt,r} - h_{tt,r} - \eta_{tt,rr}\Delta r_p - h_{tt,rr}\Delta r_p)] (\dot{r}_p + \Delta\dot{r}_p)$$

$$\begin{aligned} -2\Gamma_{tr}^r \left(\frac{dr}{dt} \right) &= -[g^{rr}g_{rr,t} + g^{rt}g_{tt,r}] (\dot{r}_p + \Delta\dot{r}_p) \simeq \\ &-[(\eta^{rr} + h^{rr} + \eta_{,r}^{rr}\Delta r_p + h_{,r}^{rr}\Delta r_p) (h_{rr,t} + h_{rr,tr}\Delta r_p) + \\ &(h^{rt} + h_{,r}^{rt}\Delta r_p) (\eta_{tt,r} + h_{tt,r} + \eta_{tt,rr}\Delta r_p + h_{tt,rr}\Delta r_p)] (\dot{r}_p + \Delta\dot{r}_p) \quad (27) \end{aligned}$$

$$\begin{aligned} -\Gamma_{tt}^r &= -\frac{1}{2} [g^{rr}(2g_{rt,t} - g_{tt,r}) + g^{rt}g_{tt,t}] \simeq \\ &-\frac{1}{2} [(\eta^{rr} + h^{rr} + \eta_{,r}^{rr}\Delta r_p + h_{,r}^{rr}\Delta r_p) (2h_{rt,t} + 2h_{rt,tr}\Delta r_p - \eta_{tt,r} - h_{tt,r} - \eta_{tt,rr}\Delta r_p - h_{tt,rr}\Delta r_p) \\ &+ (h^{rt} + h_{,r}^{rt}\Delta r_p) (h_{tt,t} + h_{tt,rr}\Delta r_p)] \quad (28) \end{aligned}$$

The development of eqs.(23 - 28) produces the coefficients vertically listed in *Tab. 1*. The lines, hierarchically developed in horizontal rows, show the α_i ($i = 1, \dots, 8$) terms of eq.(4). The e.g. α_1 term is given by:

$$\alpha_1 = \eta_{,r}^{tt} \eta_{tt,r} \dot{r}_p^2 + \eta^{tt} \eta_{tt,rr} \dot{r}_p^2 - \frac{1}{2} \eta_{,r}^{rr} \eta_{rr,r} \dot{r}_p^2 - \frac{1}{2} \eta^{rr} \eta_{rr,rr} \dot{r}_p^2 + \frac{1}{2} \eta_{,r}^{rr} \eta_{tt,r} + \frac{1}{2} \eta^{rr} \eta_{tt,rr} \quad (29)$$

and results into eq.(31). It's obvious that a similar development takes place for all α terms. But for the numerical estimate of $\alpha_{6,7,8}$ terms, simulation of the radial fall in time domain is required [19]. The simulation is to provide wavefunctions and thus metric perturbations. Here below all α terms are shown (Regge-Wheeler gauge $H_0 = H_2$):

$$\alpha_0 = -\frac{M}{r} \left(\frac{r-2M}{r^2} - \frac{3M}{r-2M} \dot{r}_p^2 \right) \quad (30)$$

$$\alpha_1 = -\frac{2M}{r^2} \left[\frac{3M}{r^2} - \frac{1}{r} + \frac{3(r-M)}{(r-2M)^2} \dot{r}_p^2 \right] \quad (31)$$

$$\alpha_2 = \frac{6M}{r(r-2M)} \dot{r}_p^2 \quad (32)$$

$$\alpha_3 = \frac{4M^2}{r^3} \left[\frac{1}{r} + \frac{r-4M}{(r-2M)^3} \dot{r}_p^2 \right] \quad (33)$$

$$\alpha_4 = \frac{3M}{r(r-2M)} \quad (34)$$

$$\alpha_5 = -\frac{4M}{r(r-2M)} \left[\frac{M}{r(r-2M)} + \frac{2}{r} + \frac{1}{r-2M} \right] \dot{r}_p \quad (35)$$

$$\begin{aligned} \alpha_6 = & \frac{1}{r-2M} \left[\frac{r^2}{2(r-2M)} \dot{H}_0 - \frac{M}{r-2M} H_1 - r H_1' \right] \dot{r}_p^3 - \frac{3}{2} H_0' \dot{r}_p^2 - 3 \left(\frac{1}{2} \dot{H}_0 - \frac{M}{r^2} H_1 \right) \dot{r}_p \\ & + \frac{r-2M}{r} \left(\frac{2M}{r^2} H_0 - \frac{1}{2} \frac{r-2M}{r} H_0' - \dot{H}_1 \right) \end{aligned} \quad (36)$$

$$\begin{aligned} \alpha_7 = & -\frac{1}{r-2M} \left[\frac{2Mr}{(r-2M)^2} \dot{H}_0 - \frac{1}{2} \frac{r^2}{r-2M} \dot{H}_0' - \frac{2M}{(r-2M)^2} H_1 - \frac{M}{r-2M} H_1' + r H_1'' \right] \dot{r}_p^3 \\ & - \frac{3}{2} H_0'' \dot{r}_p^2 - \frac{3}{2} \left(\dot{H}_0' + \frac{4M}{r^3} H_1 - \frac{2M}{r^2} H_1' \right) \dot{r}_p \end{aligned} \quad (37)$$

$$\begin{aligned} & -\frac{1}{r} \left[\frac{4M(r-3M)}{r^3} H_0 - \frac{4M(r-2M)}{r^2} H_0' - \frac{1}{2} \frac{(r-2M)^2}{r} H_0'' + \frac{2M}{r} \dot{H}_1 + (r-2M) \dot{H}_1' \right] \\ \alpha_8 = & \frac{3}{r-2M} \left[\frac{1}{2} \frac{r^2}{r-2M} \dot{H}_0 - \frac{M}{r-2M} H_1 - r H_1' \right] \dot{r}_p^2 \\ & - 3 H_0' \dot{r}_p - \frac{3}{2} \dot{H}_0 + \frac{3M}{r^2} H_1 \end{aligned} \quad (38)$$

